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The Lorentz reciprocal theorem for micropolar fluids

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Abstract. A generalization of the Lorentz reciprocal theorem is developed for the creeping flow of micropolar fluids in which the continuum equations involve both the velocity and the internal spin vector fields. In this case, the stress tensor is generally not symmetric and conservation laws for both linear and angular momentum are needed in order to describe the dynamics of the fluid continuum. This necessitates the introduction of constitutive equations for the antisymmetric part of the stress tensor and the so-called couple-stress in the medium as well. The reciprocal theorem, derived herein in the limit of negligible inertia and without external body forces and couples, provides a general integral relationship between the velocity, spin, stress and couple-stress fields of two otherwise unrelated micropolar flow fields occurring in the same fluid domain.

1. Introduction

In his original paper, whose publication is commemorated by the appearance of this centennial volume, Lorentz [1] derived a reciprocal theorem governing the slow viscous flow of incompressible, Newtonian fluids. This theorem subsequently found wide use in applications, especially to problems pertaining to general low Reynolds number flows occurring in fluid/particle systems, such as suspensions, dispersions, emulsions, porous media and clusters composed of a finite number of particles — including one as a lower limit. Contributions in this context prior to 1965 are documented in the textbook by Happel and Brenner [2]. Many developments have occurred since, and some of these are reviewed by other contributors to the present volume. Our contribution is concerned with deriving a generalization of the Lorentz (linear momentum) reciprocal theorem to the case where angular momentum is sensible too, as embodied in the existence of antisymmetric and couple stresses. In this sense, our analysis extends Lorentz's analysis from structureless to internally structured fluid continua in which spin plays a kinematical role comparable to that played by the velocity in classical problems.

2. Micropolar fluids

In micropolar or structured fluids [3–5], in which the internal microstructure may possess its own spin (and therefore angular momentum), the stress tensor quantifying the continuum dynamics is not necessarily symmetric. An example of such a fluid is the well-known magnetic liquid ('ferrofluid') which consists of a stabilized colloidal suspension of Brownian magnetic particles in a nonmagnetic liquid host [6]. In such fluids, Cauchy's moment of momentum principle provides an independent equation which must be solved simultaneously with the equations of conservation of mass and linear momentum. The complete set of conservation laws thus [3,4] consists of the continuity equation

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0, \tag{1}$$

together with Cauchy's laws of linear and angular momentum:

$$\rho\frac{D\mathbf{v}}{Dt} = \nabla\cdot\mathbf{P} + \rho\mathbf{F}^e, \tag{2}$$

$$\rho k^2\frac{D\boldsymbol{\omega}}{Dt} = \nabla\cdot\mathbf{C} + \mathbf{P}_\times + \rho\mathbf{G}^e. \tag{3}$$

Here, $\mathbf{v}$ and $\boldsymbol{\omega}$ respectively denote the velocity and spin vector fields existing at each point $\mathbf{x}$ of the fluid continuum, ρ is the mass density, k the radius of gyration field defined such that $k^2\boldsymbol{\omega}$ is the specific (i.e., per unit mass) internal angular momentum density. The vector fields $\mathbf{F}^e$ and $\mathbf{G}^e$ represent the respective specific external body force and body couple densities, whereas $\mathbf{P}$ and $\mathbf{C}$ are the stress and couple-stress dyadics. Moreover, $\mathbf{P}_\times \equiv -\boldsymbol{\epsilon}:\mathbf{P} = -\boldsymbol{\epsilon}:\mathbf{P}^a$ is the pseudovector-equivalent of the antisymmetric portion, $\mathbf{P}^a \equiv (\mathbf{P}-\mathbf{P}^\dagger)/2$, of the stress tensor, in which $\boldsymbol{\epsilon}$ is the unit alternating pseudoisotropic triadic (whose Cartesian tensor equivalent is the permutation symbol, ϵ_{ijk}). Equivalently, $\mathbf{P}^a = (\boldsymbol{\epsilon}\cdot\mathbf{P}_\times)/2$. The superscript '$\dagger$' designates the transpose operator. As usual, $D/Dt \equiv \partial/\partial t + \mathbf{v}\cdot\nabla$ denotes the substantial time derivative.

For a 'Newtonian-like' micropolar fluid the linear constitutive equations relating the stresses to the kinematical velocity and spin fields are respectively given by [7]

$$\mathbf{P} = -p\mathbf{I} + \kappa\mathbf{I}(\nabla\cdot\mathbf{v}) + 2\mu\mathbf{E}_v + \tfrac{1}{2}\boldsymbol{\epsilon}\cdot\mathbf{P}_\times, \tag{4}$$

$$\mathbf{C} = \nu_1\mathbf{I}(\nabla\cdot\boldsymbol{\omega}) + 2\nu_2\mathbf{E}_\omega, \tag{5}$$

in which

$$\mathbf{P}_\times = \zeta(\tfrac{1}{2}\nabla\times\mathbf{v} - \boldsymbol{\omega}), \tag{6}$$

and with p the thermodynamic pressure. The symmetric, traceless rate-of-strain and rate-of-spin-strain dyadics are defined by

$$\mathbf{E}_v = \tfrac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger] - \tfrac{1}{3}\mathbf{I}(\nabla\cdot\mathbf{v}), \tag{7}$$

$$\mathbf{E}_\omega = \tfrac{1}{2}[\nabla\boldsymbol{\omega} + (\nabla\boldsymbol{\omega})^\dagger] - \tfrac{1}{3}\mathbf{I}(\nabla\cdot\boldsymbol{\omega}). \tag{8}$$

The material scalar pairs (κ, μ) and (ν_1, ν_2) represent the respective (dilatational, shear) classical and spin viscosities, whereas ζ is the so-called vortex-viscosity. Additionally, $\mathbf{I}$ is the unit isotropic tensor and $(\nabla\times\mathbf{v})/2$ is one-half the vorticity, the latter providing the local rate of rotation of a fluid element.

2.1. Energy and Thermodynamic Considerations

In order to obtain the equation of conservation of energy for a micropolar fluid, one recognizes that the specific density $\mathcal{E}$ is given by

$$\mathcal{E} = \mathcal{U} + \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + \tfrac{1}{2}k^2\boldsymbol{\omega}\cdot\boldsymbol{\omega},$$

consisting of internal (U) as well as translational and rotational kinetic energies. The conservation of energy for an arbitrary material volume $V_m(t)$ with bounding surface $S_m(t)$ is thus given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_m(t)} \rho\mathcal{E}\,\mathrm{d}V = \int_{V_m(t)} \rho(\mathbf{F}^e\cdot\mathbf{v} + \mathbf{G}^e\cdot\boldsymbol{\omega})\,\mathrm{d}V + \int_{S_m(t)} [\mathbf{n}\cdot(\mathbf{P}\cdot\mathbf{v} + \mathbf{C}\cdot\boldsymbol{\omega}) - \mathbf{n}\cdot\mathbf{q}]\,\mathrm{d}S. \tag{9}$$

Here, $\mathbf{n}$ is the outward unit normal at the surface and $\mathbf{q}$ represents the flux of heat by thermal conduction. The volume integral on the right-hand-side (RHS) provides the work done per unit time by the external body forces and couples acting upon the material volume, whereas the surface integral provides the rate at which work is being done by the stresses and couple-stresses acting across the bounding surfaces, together with the total rate of heat flow across the moving surface. With the aid of Reynolds transport theorem and Gauss' divergence theorem, the differential form of the above energy equation is found to be

$$\rho\frac{D\mathcal{E}}{Dt} = \rho\mathbf{F}^e\cdot\mathbf{v} + \rho\mathbf{G}^e\cdot\boldsymbol{\omega} + \boldsymbol{\nabla}\cdot(\mathbf{P}\cdot\mathbf{v}) + \boldsymbol{\nabla}\cdot(\mathbf{C}\cdot\boldsymbol{\omega}) - \boldsymbol{\nabla}\cdot\mathbf{q}. \tag{10}$$

Upon dot multiplying (2) and (3) respectively by $\mathbf{v}$ and $\boldsymbol{\omega}$ and using the results to simplify (10) one finds that

$$\rho\frac{D\mathcal{U}}{Dt} = \mathbf{P}^\dagger : \boldsymbol{\nabla}\mathbf{v} + \mathbf{C}^\dagger : \boldsymbol{\nabla}\boldsymbol{\omega} - \mathbf{P}_\times\cdot\boldsymbol{\omega} - \boldsymbol{\nabla}\cdot\mathbf{q}. \tag{11}$$

It is often convenient to define a 'mean' pressure by $\bar{p} = -(\mathbf{I}:\mathbf{P})/3$. Upon combining this definition with constitutive equation (4), one thus finds the relation between the mean and thermodynamic pressures to be

$$\bar{p} = p - \kappa\boldsymbol{\nabla}\cdot\mathbf{v}. \tag{12}$$

To simplify further, constitutive Eqs. (4) and (5) are employed and use is made of the general identity that, if $\mathbf{S}'$ and $\mathbf{S}''$ are arbitrary symmetric dyadics while $\mathbf{A}'$ and $\mathbf{A}''$ are antisymmetric (with $\mathbf{A}'_\times = -\boldsymbol{\epsilon}:\mathbf{A}'$ and $\mathbf{A}''_\times = -\boldsymbol{\epsilon}:\mathbf{A}''$ their pseudovector-equivalents),

$$(\mathbf{S}' + \mathbf{A}'):(\mathbf{S}'' + \mathbf{A}'') = \mathbf{S}':\mathbf{S}'' + \mathbf{A}':\mathbf{A}'' = \mathbf{S}':\mathbf{S}'' - \tfrac{1}{2}\mathbf{A}'_\times\cdot\mathbf{A}''_\times,$$

to obtain

$$\mathbf{P}^\dagger : \boldsymbol{\nabla}\mathbf{v} = -\bar{p}(\boldsymbol{\nabla}\cdot\mathbf{v}) + 2\mu\mathbf{E}_v : \mathbf{E}_v + \mathbf{P}_\times\cdot(\tfrac{1}{2}\boldsymbol{\nabla}\times\mathbf{v}), \tag{13}$$

$$\mathbf{C}^\dagger : \boldsymbol{\nabla}\boldsymbol{\omega} = \nu_1(\boldsymbol{\nabla}\cdot\boldsymbol{\omega})^2 + 2\nu_2\mathbf{E}_\omega : \mathbf{E}_\omega. \tag{14}$$

Substitution of these results in the internal energy Eq. (11) allows the latter to be written as

$$\rho\frac{D\mathcal{U}}{Dt} = -\bar{p}\boldsymbol{\nabla}\cdot\mathbf{v} + 2\mu\mathbf{E}_v : \mathbf{E}_v + \mathbf{P}_\times\cdot(\tfrac{1}{2}\boldsymbol{\nabla}\times\mathbf{v} - \boldsymbol{\omega}) + \nu_1(\boldsymbol{\nabla}\cdot\boldsymbol{\omega})^2 + 2\nu_2\mathbf{E}_\omega : \mathbf{E}_\omega - \boldsymbol{\nabla}\cdot\mathbf{q}. \tag{15}$$

Now, according to the combined first and second laws of thermodynamics, we have that

$$\mathrm{d}\mathcal{U} = T\,\mathrm{d}\mathcal{S} - p\,\mathrm{d}\mathcal{V},$$

where $\mathcal{S}$ is the specific entropy and $\mathcal{V}$ the specific volume, the latter being related to the density by $\mathcal{V} = 1/\rho$. In conjunction with the continuity Eq. (1), the above thermodynamic relation is equivalent to

$$\frac{D\mathcal{U}}{Dt} = T\frac{D\mathcal{S}}{Dt} - \frac{p}{\rho}\nabla\cdot\mathbf{v}.$$

Now, the generic entropy balance Eq. (8) is

$$\rho\frac{D\mathcal{S}}{Dt} + \nabla\cdot\mathbf{J}_s = \sigma, \tag{16}$$

where $\mathbf{J}_s \equiv \mathbf{q}/T$ is the entropy flux density vector and σ is the volumetric rate of entropy production [8]. Upon using the preceding relations in conjunction with Fourier's law of heat conduction

$$\mathbf{q} = -k\nabla T,$$

[with k the thermal conductivity, not to be confused with the radius of gyration appearing in Eq. (3)] and the constitutive Eq. (6) for the pseudovector of the antisymmetric stress, we thereby obtain

$$T\sigma = \frac{k}{T}|\nabla T|^2 + \kappa(\nabla\cdot\mathbf{v})^2 + 2\mu\mathbf{E}_v : \mathbf{E}_v + \nu_1(\nabla\cdot\boldsymbol{\omega})^2 + 2\nu_2\mathbf{E}_\omega : \mathbf{E}_\omega + \zeta|\tfrac{1}{2}\nabla\times\mathbf{v} - \boldsymbol{\omega}|^2. \tag{17}$$

As σ is required to be non-negative as a consequence of irreversibility, and to vanish if and only if all the gradients are identically zero, it follows that k together with the viscosity coefficients $\kappa, \mu, \nu_1, \nu_2$ and ζ appearing in (17) are all positive.

3. The reciprocal theorem

Consider the 'creeping flow' of an incompressible micropolar fluid, for which the inertial terms appearing on the left-hand-sides (LHS) of Eqs. (2) and (3) are negligible. In the further absence of external body forces and couples, the mass, linear and angular momentum Eqs. (1)–(3), respectively, reduce to the forms:

$$\nabla\cdot\mathbf{v} = 0, \tag{18}$$

$$\nabla\cdot\mathbf{P} = \mathbf{0}, \tag{19}$$

$$\nabla\cdot\mathbf{C} + \mathbf{P}_\times = \mathbf{0}, \tag{20}$$

although the constitutive equations given by (4)–(8) remain the same.

To extend the classical Lorentz reciprocal theorem to the case of a micropolar fluid, consider two flow fields $(\mathbf{v}', \boldsymbol{\omega}')$ and $(\mathbf{v}'', \boldsymbol{\omega}'')$ occurring within the same micropolar fluid and same geometrical domain, respectively designated by a prime and a double prime. Both satisfy the field Eqs. (18)–(20) within the specified fluid domain, but with different boundary conditions on the surfaces bounding this domain. Prompted by the forms of the work-related, third and fourth terms appearing on the RHS of (10), one calculates

$$\nabla\cdot(\mathbf{P}'\cdot\mathbf{v}'') = (\mathbf{P}')^\dagger : \nabla\mathbf{v}'' = 2\mu\mathbf{E}'_v : \mathbf{E}''_v + \mathbf{P}'_\times\cdot(\tfrac{1}{2}\nabla\times\mathbf{v}''), \tag{21}$$

$$\nabla\cdot(\mathbf{C}'\cdot\boldsymbol{\omega}'') = -\mathbf{P}'_{\times}\cdot\boldsymbol{\omega}'' + (\mathbf{C}')^{\dagger} : \nabla\boldsymbol{\omega}'' = -\mathbf{P}'_{\times}\cdot\boldsymbol{\omega}'' + \nu_1(\nabla\cdot\boldsymbol{\omega}')(\nabla\cdot\boldsymbol{\omega}'') + 2\nu_2\mathbf{E}'_{\omega} : \mathbf{E}''_{\omega}. \tag{22}$$

Upon summing both sides of the last two equations and making use of the constitutive relation (6) for $\mathbf{P}'_{\times}$, it is found that the terms on the RHS of the resulting sum are invariant under interchange of the prime and double prime affixes. Therefore, the same is true of the LHS, whereupon it follows that

$$\nabla\cdot(\mathbf{P}'\cdot\mathbf{v}'' + \mathbf{C}'\cdot\boldsymbol{\omega}'') = \nabla\cdot(\mathbf{P}''\cdot\mathbf{v}' + \mathbf{C}''\cdot\boldsymbol{\omega}'). \tag{23}$$

Upon integrating both sides of the latter over the entire volume occupied by the fluid and making use of the divergence theorem, the final form of the generalized Lorentz reciprocal theorem for micropolar fluids is found to be

$$\int_S \mathbf{n}\cdot(\mathbf{P}'\cdot\mathbf{v}'' + \mathbf{C}'\cdot\boldsymbol{\omega}'')\,\mathrm{d}S = \int_S \mathbf{n}\cdot(\mathbf{P}''\cdot\mathbf{v}' + \mathbf{C}''\cdot\boldsymbol{\omega}')\,\mathrm{d}S, \tag{24}$$

in which S represents the closed surface bounding the fluid domain internally (including a possible surface at 'infinity') and $\mathbf{n}$ is the unit normal pointing into the fluid on the bounding surface S.

3.1. Application: Symmetry of the Micropolar Hydrodynamic Resistance Matrix

As an application of this general theorem, it will now be shown that the resistance "matrix" which relates the hydrodynamic force and torque on a rigid body to its translational and angular velocities in an unbounded micropolar fluid which is at rest at infinity is both symmetric and positive-definite. Thus, consider the problem of translation and rotation of a rigid body of arbitrary shape through an otherwise quiescent (and spinless) micropolar fluid under creeping flow conditions for which the governing equations are given by (18)–(20), together with the constitutive relations (4)–(8). The standard, no-slip boundary conditions [9] imposed on the fluid velocity and spin fields are thus given by

$$\mathbf{v} = \mathbf{U} + \boldsymbol{\Omega}\times\mathbf{x}, \quad \boldsymbol{\omega} = \boldsymbol{\Omega}, \quad \text{on} \quad S_p, \tag{25}$$

where S_p denotes the surface of the rigid particle. In the above, $\boldsymbol{\Omega}$ is the angular velocity of the rigid body, $\mathbf{U}$ is the translational velocity of some locator-point rigidly attached to the body, and $\mathbf{x}$ is the position vector measured relative to the same locator-point. Additionally, we require that the fields $\mathbf{v}$ and $\boldsymbol{\omega}$ vanish at infinity. As in the case of conventional, nonpolar fluids one can show that the rate of decay in the fluid of these fields at infinity is such that $\mathbf{v} = O(|\mathbf{x}|^{-1})$ and $\boldsymbol{\omega} = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$, so that only the surface S_p of the particle contributes to the surface integrals appearing in (25). In such circumstances the reciprocal theorem (25) reduces to one involving integrals over S_p only.

Since the governing equations and boundary conditions are linear, the disturbance pressure, velocity and spin fields are necessarily linear in the vectors $\mathbf{U}$ and $\boldsymbol{\Omega}$, as too are the stress and couple-stress fields. As a result, the hydrodynamic force and torque (the latter about the particle locator-point) exerted by the fluid on the body, namely

$$\mathbf{F} = \int_{S_p} \mathbf{n}\cdot\mathbf{P}\,\mathrm{d}S, \tag{26}$$

$$\mathbf{T} = \int_{S_p} [\mathbf{x}\times(\mathbf{n}\cdot\mathbf{P}) + \mathbf{n}\cdot\mathbf{C}]\,\mathrm{d}S, \tag{27}$$

are also linear in $\mathbf{U}$ and $\boldsymbol{\Omega}$. As such, proceeding as in the classical case [2] where angular momentum effects are absent, evaluation of the integrals in (26) and (27) will produce a general linear relation of the (hybrid, vector/partitioned-matrix) form

$$\begin{bmatrix}\mathbf{F}\\ \mathbf{T}\end{bmatrix} = -\begin{bmatrix}\mathbf{R}_{FU} & \mathbf{R}_{F\Omega}\\ \mathbf{R}_{TU} & \mathbf{R}_{T\Omega}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{U}\\ \boldsymbol{\Omega}\end{bmatrix}, \tag{28}$$

in which the $\mathbf{R}$ entries within the resistance matrix on the RHS are each dyadics (3×3 matrices) which are independent of $\mathbf{U}$ and $\boldsymbol{\Omega}$. They depend only upon the size and shape of the body (including — with the exception of the $\mathbf{R}_{FU}$ entry — the choice of particle locator point) in addition to depending upon the various viscosity coefficients characterizing the fluid.

To prove that the resistance matrix is symmetric, suppose that the given rigid body undergoes two different translational and rotational motions, respectively characterized by $(\mathbf{U}', \boldsymbol{\Omega}')$ and $(\mathbf{U}'', \boldsymbol{\Omega}'')$. The resulting hydrodynamic force and torque on the body in each case are respectively given by (28), with primes or double-primes affixed to $\mathbf{F}, \mathbf{T}, \mathbf{U}$ and $\boldsymbol{\Omega}$, but not to the resistance tensors $\mathbf{R}$ (since the latter do not depend on the translational and angular velocities of the body). For the present flow, which is quiescent far away from the rigid body, the only surface integrals which contribute to the generalized Lorentz reciprocal theorem (24) lie on the body surface S_p. With the aid of boundary conditions (25), the LHS of (24) can be written as

$$\int_{S_p} [\mathbf{n}\cdot\mathbf{P}'\cdot(\mathbf{U}'' + \boldsymbol{\Omega}\times\mathbf{x}) + \mathbf{n}\cdot\mathbf{C}'\cdot\boldsymbol{\Omega}'']\,\mathrm{d}S = \int_{S_p} \{(\mathbf{n}\cdot\mathbf{P}')\cdot\mathbf{U}'' + [\mathbf{x}\times(\mathbf{n}\cdot\mathbf{P}') + \mathbf{n}\cdot\mathbf{C}']\cdot\boldsymbol{\Omega}''\}\,\mathrm{d}S,$$

which is the same as

$$\mathbf{F}'\cdot\mathbf{U}'' + \mathbf{T}'\cdot\boldsymbol{\Omega}'' \equiv [\mathbf{U}''\quad \boldsymbol{\Omega}'']\cdot\begin{bmatrix}\mathbf{F}'\\ \mathbf{T}'\end{bmatrix}.$$

The RHS of (24) can be computed in a similar manner, resulting in an identical expression, but with primes and double-primes interchanged. Upon making use of expression (28) for the forces and torques, the reciprocal theorem thus shows that

$$[\mathbf{U}''\quad \boldsymbol{\Omega}'']\cdot\begin{bmatrix}\mathbf{R}_{FU} & \mathbf{R}_{F\Omega}\\ \mathbf{R}_{TU} & \mathbf{R}_{T\Omega}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{U}'\\ \boldsymbol{\Omega}'\end{bmatrix} = [\mathbf{U}'\quad \boldsymbol{\Omega}']\cdot\begin{bmatrix}\mathbf{R}_{FU} & \mathbf{R}_{F\Omega}\\ \mathbf{R}_{TU} & \mathbf{R}_{T\Omega}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{U}''\\ \boldsymbol{\Omega}''\end{bmatrix}. \tag{29}$$

Being a scalar, each side of the preceding equation is equal to its own transpose. Thus, upon taking the transpose of one side of the above equation and canceling the arbitrary translational-angular velocity vector pair which appears symmetrically on both sides of the resulting equation, one establishes that

$$\begin{bmatrix}\mathbf{R}_{FU} & \mathbf{R}_{F\Omega}\\ \mathbf{R}_{TU} & \mathbf{R}_{T\Omega}\end{bmatrix} = \begin{bmatrix}\mathbf{R}_{FU} & \mathbf{R}_{F\Omega}\\ \mathbf{R}_{TU} & \mathbf{R}_{T\Omega}\end{bmatrix}^{\dagger}, \tag{30}$$

completing the proof of symmetry of the resistance matrix. The corresponding relations for the individual resistance entries appearing in this matrix are of the respective forms

$$\mathbf{R}_{FU} = \mathbf{R}^{\dagger}_{FU},\quad \mathbf{R}_{T\Omega} = \mathbf{R}^{\dagger}_{T\Omega},\quad \mathbf{R}_{F\Omega} = \mathbf{R}^{\dagger}_{TU}. \tag{31}$$

In the last of this trio of equations, note that the resistance dyadic $\mathbf{R}_{F\Omega}$ which relates the hydrodynamic force on the body to its angular velocity is equal to the transpose of the resistance dyadic $\mathbf{R}_{TU}$, which relates the torque on the body to its translational velocity.

To prove that the resistance tensor is positive-definite, we observe that $-\mathbf{F}\cdot\mathbf{U} - \mathbf{T}\cdot\boldsymbol{\Omega}$ represents the rate at which the body does work on the surrounding fluid. Since this rate must be nonnegative, one can write

$$[\mathbf{U} \quad \boldsymbol{\Omega}]\cdot\begin{bmatrix}\mathbf{R}_{FU} & \mathbf{R}_{F\Omega}\\ \mathbf{R}_{TU} & \mathbf{R}_{T\Omega}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{U}\\ \boldsymbol{\Omega}\end{bmatrix} \geqslant 0, \tag{32}$$

in which equality holds only when both $\mathbf{U} = \mathbf{0}$ and $\boldsymbol{\Omega} = \mathbf{0}$. Since this inequality must hold for arbitrary choices of the vectors $\mathbf{U}$ and $\boldsymbol{\Omega}$, the resistance matrix is seen to be positive-definite. This positivity extends to the 'direct' resistance dyadics $\mathbf{R}_{FU}$ and $\mathbf{R}_{T\Omega}$, but not to the 'coupling' dyadic $\mathbf{R}_{F\Omega}$ or its kinetic equivalent $\mathbf{R}_{TU}$.

Questions of the origin-dependence of these three hydrodynamic resistance dyadics, including the existence of a 'center of reaction' (at which point $\mathbf{R}_{F\Omega} = \mathbf{R}^{\dagger}_{F\Omega}$), simplifications afforded by the geometric symmetries of particles, and other similar issues are essentially identical to comparable issues existing for the classical case [2], where angular momentum considerations are absent (corresponding to ζ, ν_1 and ν_2 being identically zero). In this same vein, following well-established paths existing for the classical case, derivative relations — such as extensions of Faxén's laws for the hydrodynamic resistances of spherical and nonspherical particles suspended in arbitrary fields of flow — can be generalized to include internal spin effects. In the latter context, it should be noted that detailed solutions of the basic micropolar creeping flow equations (18)–(20) subject to the boundary conditions (25) already exist for spheres and some other body shapes, such as spheroids [10–13].

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